

The Janson inequalities for general up-sets

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Abstract

Janson and Janson, Łuczak and Ruciński proved several inequalities for the lower tail of the distribution of the number of events that hold, when all the events are up-sets (increasing events) of a special form – each event is the product of some subset of a single set of independent Bernoulli random variables (i.e., a principal up-set). We show that these inequalities in fact hold for arbitrary up-sets, by modifying existing proofs to use only positive correlation, avoiding the need to assume positive correlation conditioned on one of the events.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{I} \subseteq \mathcal{F}$ a collection of events with the following properties:

$$A, B \in \mathcal{I} \implies \mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B) \quad (1)$$

and

$$A, B \in \mathcal{I} \implies A \cap B \in \mathcal{I} \text{ and } A \cup B \in \mathcal{I}. \quad (2)$$

Of course, the standard and most important example is when $\Omega = \{0, 1\}^X$ is a product probability space (with product measure), and \mathcal{I} is the collection of increasing events, i.e., events A such that $\omega \in A$ and $\omega \leq \omega'$ pointwise imply $\omega' \in A$. Then (1) is simply Harris's Lemma [4] (also known as Kleitman's Lemma). There are other examples, such as increasing events in random cluster models with parameter $q \geq 1$; see [3].

Let $A_1, \dots, A_k \in \mathcal{I}$, write I_i for the indicator function of A_i , and set

$$X = \sum_i I_i, \quad \mu = \mathbb{E}(X) = \sum_i \mathbb{P}(A_i)$$

and

$$\Delta = \sum_i \sum_{j \sim i} \mathbb{P}(A_i \cap A_j),$$

where we write $i \sim j$ if $i \neq j$ and A_i and A_j are dependent. (Note that we sum over *ordered* pairs, and exclude the term $i = j$. These conventions are not universal!)

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Theorem 1. *Under the conditions above we have*

$$\mathbb{P}(X = 0) \leq \exp(-\mu + \Delta/2), \quad (3)$$

and, for any $0 \leq t \leq \mu$,

$$\mathbb{P}(X \leq \mu - t) \leq \exp\left(-\frac{\phi(-t/\mu)\mu^2}{\mu + \Delta}\right) \leq \exp\left(-\frac{t^2}{2(\mu + \Delta)}\right), \quad (4)$$

where $\phi(x) = (1+x)\log(1+x) - x$ with $\phi(-1) = 1$.

When the events A_i are *principal* up-sets, i.e., events in a product space $\{0,1\}^X$ of the form $A_i = \{\omega : \omega_x = 1 \text{ for all } x \in \alpha_i\}$ for some $\alpha_i \subseteq X$, the first inequality in (4) is the well known Janson inequality [5]; for various convenient weaker forms see [7]. Under the same assumptions, (3) was proved earlier by Janson, Łuczak and Ruciński [6]. We shall prove Theorem 1 by modifying the proofs of these inequalities to avoid applying Harris's Lemma to the conditional measure $\mathbb{P}(\cdot \mid A_i)$.

Proof. We begin with a simple observation that, in the standard setting, follows from the equality conditions in Harris's Lemma. Indeed, we claim that, for each i ,

$$A_i \text{ is independent of the set } \mathcal{S}_i = \{A_j : j \neq i, j \not\sim i\}. \quad (5)$$

To see this, note first that if $A, B, C \in \mathcal{I}$ and A is independent of B and of C , then

$$\mathbb{P}(A \cap (B \cap C)) + \mathbb{P}(A \cap (B \cup C)) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) = \mathbb{P}(A)(\mathbb{P}(B) + \mathbb{P}(C)). \quad (6)$$

Since, by (2), $B \cap C$ and $B \cup C$ are in \mathcal{I} , by (1) we have

$$\mathbb{P}(A \cap (B \cap C)) \geq \mathbb{P}(A)\mathbb{P}(B \cap C) \text{ and } \mathbb{P}(A \cap (B \cup C)) \geq \mathbb{P}(A)\mathbb{P}(B \cup C).$$

The sum of these inequalities is the equality (6), so both inequalities are equalities, and in particular A is independent of $B \cap C$. Since A_i is independent of each $A_j \in \mathcal{S}_i$ by definition, it follows inductively that A_i is independent of any intersection of events $A_j \in \mathcal{S}_i$, so A_i is independent of the set \mathcal{S}_i of events, as claimed.

To prove (3) we modify the argument given by Boppana and Spencer [2], following the presentation in [1]. Following Boppana and Spencer, set $r_i = \mathbb{P}(A_i \mid A_1^c \cap \dots \cap A_{i-1}^c)$, so $\mathbb{P}(X = 0) = \prod_{i=1}^k (1 - r_i) \leq \exp(-\sum r_i)$. It suffices to show that for each i we have

$$r_i \geq \mathbb{P}(A_i) - \sum_{j < i, j \sim i} \mathbb{P}(A_i \cap A_j), \quad (7)$$

since the sum of the final expression is exactly $\mu - \Delta/2$. Fix i . Still following [2, 1], set

$$D_0 = \bigcap_{j < i, j \not\sim i} A_j^c \quad \text{and} \quad D_1 = \bigcap_{j < i, j \sim i} A_j^c,$$

so (by (5)) A_i is independent of D_0 . Then

$$r_i = \mathbb{P}(A_i \mid D_0 \cap D_1) = \frac{\mathbb{P}(A_i \cap D_0 \cap D_1)}{\mathbb{P}(D_0 \cap D_1)} \geq \frac{\mathbb{P}(A_i \cap D_0 \cap D_1)}{\mathbb{P}(D_0)} = \mathbb{P}(A_i \cap D_1 \mid D_0). \quad (8)$$

(We may assume that $\mathbb{P}(D_0 \cap D_1) > 0$, since otherwise $\mathbb{P}(X = 0) = 0$.) At this point the original argument involves writing the final probability as $\mathbb{P}(A_i \mid D_0)\mathbb{P}(D_1 \mid A_i \cap D_0)$. Instead, we simply write

$$\mathbb{P}(A_i \cap D_1 \mid D_0) = \mathbb{P}(A_i \mid D_0) - \mathbb{P}(A_i \cap D_1^c \mid D_0). \quad (9)$$

By (2), $D_1^c = \bigcup_{j < i, j \sim i} A_j \in \mathcal{I}$ (it is an up-set), so $A_i \cap D_1^c \in \mathcal{I}$. Since $D_0^c \in \mathcal{I}$ (D_0 is a down-set), (1) and the union bound give

$$\mathbb{P}(A_i \cap D_1^c \mid D_0) \leq \mathbb{P}(A_i \cap D_1^c) = \mathbb{P}\left(A_i \cap \bigcup_{j < i, j \sim i} A_j\right) \leq \sum_{j < i, j \sim i} \mathbb{P}(A_i \cap A_j). \quad (10)$$

Recalling that $\mathbb{P}(A_i \mid D_0) = \mathbb{P}(A_i)$, combining (8), (9) and (10) gives (7), completing the proof of (3).

Turning to (4), fix $1 \leq i \leq k$ and let

$$Y_i = I_i + \sum_{j \sim i} I_j \quad \text{and} \quad Z_i = \sum_{j \neq i, j \not\sim i} I_j,$$

so $X = Y_i + Z_i$, with Z_i containing the terms independent of I_i and Y_i the others (including I_i itself). In the proof of (4) given in [7], the only step in which anything is assumed about the events A_i is (2.20) on page 32, where it is shown (in our notation) that for $s \geq 0$ and each $1 \leq i \leq k$,

$$\mathbb{E}(I_i e^{-sX}) \geq \mathbb{E}(I_i e^{-sY_i}) \mathbb{E}(e^{-sZ_i}). \quad (11)$$

Proceeding much as in the proof of (7), note that

$$\frac{\mathbb{E}(I_i e^{-sX})}{\mathbb{E}(e^{-sX})} = \frac{\mathbb{E}(I_i e^{-sY_i})}{\mathbb{E}(e^{-sY_i} e^{-sZ_i})} \geq \frac{\mathbb{E}(I_i e^{-sY_i})}{\mathbb{E}(e^{-sY_i})} \frac{\mathbb{E}(e^{-sZ_i})}{\mathbb{E}(e^{-sX})}. \quad (12)$$

Also,

$$I_i e^{-sX} = I_i e^{-sY_i} e^{-sZ_i} = I_i e^{-sZ_i} - I_i e^{-sZ_i} (1 - e^{-sY_i}) = I_i g - f g,$$

where

$$f = I_i (1 - e^{-sY_i}) \quad \text{and} \quad g = e^{-sZ_i}.$$

Now from (5), I_i and Z_i are independent, so $\mathbb{E}(I_i g) = \mathbb{E}(I_i) \mathbb{E}(g)$. Also, in the up-set context, f is an increasing function. In the more general context where the A_i are in a set \mathcal{I} satisfying (1) and (2), the random variable f can be written as a sum of non-negative multiples of indicator functions of events in \mathcal{I} ; indeed, this is true of the I_j by assumption, and this property is preserved under

addition, multiplication, and the application of increasing functions. Similarly, $-g$ is an increasing function. It follows from (1) that $\mathbb{E}(f(-g)) \geq \mathbb{E}(f)\mathbb{E}(-g)$, i.e., that $\mathbb{E}(fg) \leq \mathbb{E}(f)\mathbb{E}(g)$. (This is an application of the FKG inequality in the up-set case.) Hence,

$$\mathbb{E}(I_i e^{-sX}) = \mathbb{E}(I_i g - fg) \geq \mathbb{E}(I_i)\mathbb{E}(g) - \mathbb{E}(f)\mathbb{E}(g).$$

Using (12) for the first step this gives

$$\frac{\mathbb{E}(I_i e^{-sX})}{\mathbb{E}(e^{-sX})} \geq \frac{\mathbb{E}(I_i e^{-sX})}{\mathbb{E}(g)} \geq \mathbb{E}(I_i) - \mathbb{E}(f) = \mathbb{E}(I_i - f) = \mathbb{E}(I_i e^{-sY_i}).$$

This is exactly (11), and the rest of the proof in [7] is unaltered. \square

Remark 1. We have stated two of the best-known and cleanest forms of the inequalities in Theorem 1. Let us note that other forms also hold in the present more general context. For example, the inequality (7) leads to the bound

$$\mathbb{P}(X = 0) \leq \prod_{i=1}^k (1 - \mathbb{P}(A_i)) \exp\left(\frac{1}{1-\varepsilon} \frac{\Delta}{2}\right), \quad (13)$$

where $\varepsilon = \max_i \mathbb{P}(A_i)$. This bound was given by Boppana and Spencer [2] (for $\varepsilon = 1/2$); see also [7].

Furthermore, (11) is the only step in the proof of Lemma 1 of [6] that requires any assumptions about the A_i , so we see that this result also holds in the present setting, giving (in our notation)

$$\log \mathbb{P}(X = 0) \leq - \sum_i \mathbb{E} \left(\frac{I_i}{I_i + \sum_{j \sim i} I_j} \right). \quad (14)$$

References

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